Cuaderno de notas de trabajo

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Cuaderno Sn 1

PRINCIPIO VARIACIONAL

Se trata de extremales en el espacio-tiempo llano de Minkowski.

1) $ds^2 = \Delta_{ij}\, dx^i\, dx^j$;

$$\Delta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

2) Considérense dos tensores doblemente covariantes:

g_{pq} y G_{pq};

y simétricos:

$g_{pq} = g_{qp}$ y $G_{pq} = G_{qp}$.

3) $$\tau J = \int_A^B \frac{G_{ij}\, dx^i dx^j}{\sqrt{g_{pq}\, dx^p dx^q}} \,.$$

4) $$\boxed{\begin{array}{c}\text{PRINCIPIO VARIACIONAL} \\[1em] \delta \tau J = 0 \\[1em] \delta \displaystyle\int_A^B \frac{G_{ij}\, dx^i dx^j}{\sqrt{g_{pq}\, dx^p dx^q}} = 0\end{array}}$$

¡Encontrar las extremales!

$$\boxed{\delta\{G_{ij}\, dx^i\, dx^j\}}$$

$$\frac{\partial G_{ij}}{\partial x^k}\, \delta x^k\, dx^i\, dx^j + 2 G_{ij}\, dx^i\, d(\delta x^j)$$

$$\boxed{\delta\{G_{ij}\, dx^i\, dx^j\} = \frac{\partial G_{ij}}{\partial x^k}\, \delta x^k\, dx^i\, dx^j + 2 G_{ij}\, dx^i\, d(\delta x^j)}$$

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$$\boxed{\delta\left\{g_{pq}\,dx^p\,dx^q\right\}^{-\frac{1}{2}}}$$

$$\boxed{-\frac{1}{2}\left\{g_{pq}\,dx^p dx^q\right\}^{-\frac{3}{2}}\left[\frac{\partial g_{pq}}{\partial x^r}\,\delta x^r dx^p dx^q + 2g_{pq}\,dx^p\,d(\delta x^q)\right]}$$

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$$\delta\left\{\frac{G_{ij}\,dx^i\,dx^j}{\sqrt{g_{pq}\,dx^p\,dx^q}}\right\}$$

$$\{g_{pq}\,dx^p dx^q\}^{-\frac{1}{2}}\left[\frac{\partial G_{ij}}{\partial x^k}\delta x^k dx^i dx^j + 2\,G_{ij}\,dx^i\,d(\delta x^j)\right]$$
$$-\frac{1}{2}G_{ij}\,dx^i dx^j\{g_{pq}\,dx^p dx^q\}^{-\frac{3}{2}}\left[\frac{\partial g_{pq}}{\partial x^r}\delta x^r dx^p dx^q + 2g_{pq}\,dx^p\,d(\delta x^q)\right]$$

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$$\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{1}{2}}\left[\frac{\partial G_{ij}}{\partial x^k}\delta x^k\frac{dx^i}{ds}\frac{dx^j}{ds}ds + 2\,G_{ij}\frac{dx^i}{ds}\,d(\delta x^j)\right]$$
$$-\frac{1}{2}G_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}}\left[\frac{\partial g_{pq}}{\partial x^r}\delta x^r\frac{dx^p}{ds}\frac{dx^q}{ds}ds + 2\,g_{pq}\frac{dx^p}{ds}\,d(\delta x^q)\right]$$

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$$\delta \left\{ \frac{G_{ij}\, dx^i\, dx^j}{\sqrt{g_{pq}\, dx^p\, dx^q}} \right\}$$

$$\left[\sqrt{g_{pq}\,dx^p\,dx^q}\,\right]$$

$$\frac{\dfrac{\partial G_{ij}}{\partial x^{\alpha}}\,\delta x^{\alpha}\,\dfrac{dx^i}{ds}\,\dfrac{dx^j}{ds}\,ds}{\sqrt{g_{pq}\dfrac{dx^p}{ds}\,\dfrac{dx^q}{ds}}}$$

$$-\frac{1}{2}\,\frac{G_{ij}\,\dfrac{\partial g_{pq}}{\partial x^{\alpha}}\,\delta x^{\alpha}\,\dfrac{dx^i}{ds}\,\dfrac{dx^j}{ds}\,\dfrac{dx^p}{ds}\,\dfrac{dx^q}{ds}\,ds}{\left\{g_{pq}\dfrac{dx^p}{ds}\,\dfrac{dx^q}{ds}\right\}^{3/2}}$$

$$+\,2\,\frac{G_{i\alpha}\,\dfrac{dx^i}{ds}\,d(\delta x^{\alpha})}{\sqrt{g_{pq}\dfrac{dx^p}{ds}\,\dfrac{dx^q}{ds}}}$$

$$-\,\frac{G_{ij}\,g_{p\alpha}\,\dfrac{dx^i}{ds}\,\dfrac{dx^j}{ds}\,\dfrac{dx^p}{ds}\,d(\delta x^{\alpha})}{\left\{g_{pq}\dfrac{dx^p}{ds}\,\dfrac{dx^q}{ds}\right\}^{+\frac{3}{2}}}$$

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$$\boxed{\delta\left\{\frac{G_{ij}\,dx^i\,dx^j}{\sqrt{g_{pq}\,dx^p\,dx^q}}\right\}}$$

$$\frac{\frac{\partial G_{ij}}{\partial x^\alpha}\,\delta x^\alpha\,\frac{dx^i}{ds}\,\frac{dx^j}{ds}\,ds}{\sqrt{g_{pq}\frac{dx^p}{ds}\,\frac{dx^q}{ds}}}$$

$$\frac{\partial g_{pq}}{}\,\delta x^\alpha\,\frac{dx^i}{}\,\frac{dx^j}{}\,\frac{dx^p}{}\,\frac{dx^q}{}$$

$$\boxed{+2\int_A^B \frac{G_{i\alpha}\frac{dx^i}{ds}\, d(\delta x^\alpha)}{\sqrt{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}}}}$$

B

$$-2\,\frac{\frac{\partial G_{i\alpha}}{\partial x^j}\frac{dx^i}{ds}\frac{dx^j}{ds}}{\sqrt{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}}}\,\delta x^\alpha ds - 2\,\frac{G_{i\alpha}\frac{d^2x^i}{ds^2}\,\delta x^\alpha}{\sqrt{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}}}\,ds$$

$$+\frac{G_{i\alpha}\frac{dx^i}{ds}}{\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{3/2}}\,\delta x^\alpha\,\frac{\partial g_{pq}}{\partial x^r}\frac{dx^r}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\,ds$$

$$+2\,\frac{G_{i\alpha}\frac{dx^i}{ds}}{\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{3/2}}\,\delta x^\alpha\, g_{pq}\frac{d^2x^p}{ds^2}\frac{dx^q}{ds}\,ds$$

A

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$$\boxed{+2\int_A^B \frac{G_{i\alpha}\frac{dx^i}{ds}\, d(\delta x^\alpha)}{\sqrt{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}}}}$$

$$\int^B -2\,\frac{\frac{\partial G_{i\alpha}}{\partial x^j}\frac{dx^i}{ds}\frac{dx^j}{ds}}{\sqrt{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}}}\,\delta x^\alpha ds - 2\,\frac{G_{i\alpha}\frac{d^2x^i}{ds^2}}{\sqrt{g_{pq}\frac{dx^p}{ds}\cdots}}$$

$$\boxed{-\int_A^B \frac{G_{ij}\, g_{p\alpha} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^p}{ds}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{3/2}}\, d(\delta x^\alpha)}$$

B

$$+ \frac{\frac{\partial G_{ij}}{\partial x^k}\, g_{p\alpha} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{dx^p}{ds}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{3/2}}\, \delta x^\alpha ds$$

$$+ \frac{G_{ij} \frac{\partial g_{p\alpha}}{\partial x^q} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^p}{ds} \frac{dx^q}{ds}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{3/2}}\, \delta x^\alpha ds$$

$$+ 2\, \frac{G_{ij}\, g_{p\alpha} \frac{d^2x^i}{ds^2} \frac{dx^j}{ds} \frac{dx^p}{ds}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{3/2}}\, \delta x^\alpha ds$$

$$+ \frac{G_{ij}\, g_{p\alpha} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{d^2x^p}{ds^2}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{3/2}}\, \delta x^\alpha ds$$

$$- \frac{3}{2} \frac{G_{ij}\, g_{\rho\alpha} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^\rho}{ds} \frac{\partial g_{uv}}{\partial x^w} \frac{dx^u}{ds} \frac{dx^v}{ds} \frac{dx^w}{ds}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{5/2}} \delta x^\alpha ds$$

$$- \frac{3}{\cancel{2}} \frac{G_{ij}\, g_{\rho\alpha} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^\rho}{ds}\, g_{uv} \frac{d^2 x^u}{ds^2} \frac{dx^v}{ds}}{\left\{ g_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds} \right\}^{5/2}} \delta x^\alpha ds$$

A

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PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES (Segundo miembro = 0)

Términos que contienen a las segundas derivadas de las coordenadas con respecto a s.

$$\frac{1}{\left\{ g_{\gamma\delta} \frac{dx^{\gamma}}{ds} \frac{dx^{\delta}}{ds} \right\}^{5/2}} \left[\begin{aligned} & -2 G_{\beta\alpha}\, g_{pq}\, g_{uv} \frac{dx^{p}}{ds} \frac{dx^{q}}{ds} \frac{dx^{u}}{ds} \frac{dx^{v}}{ds} \\ & +2 G_{p\alpha}\, g_{\beta q}\, g_{uv} \frac{dx^{p}}{ds} \frac{dx^{q}}{ds} \frac{dx^{u}}{ds} \frac{dx^{v}}{ds} \\ & +2 G_{p\beta}\, g_{q\alpha}\, g_{uv} \frac{dx^{p}}{ds} \frac{dx^{q}}{ds} \frac{dx^{u}}{ds} \frac{dx^{v}}{ds} \\ & -3 G_{pq}\, g_{u\alpha}\, g_{v\beta} \frac{dx^{p}}{ds} \frac{dx^{q}}{ds} \frac{dx^{u}}{ds} \frac{dx^{v}}{ds} \\ & + G_{pq}\, g_{\alpha\beta}\, g_{uv} \frac{dx^{p}}{ds} \frac{dx^{q}}{ds} \frac{dx^{u}}{ds} \frac{dx^{v}}{ds} \end{aligned} \right] \frac{d^2 x^{\beta}}{ds^2}$$

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES. (Segundo miembro = 0)

Términos que contienen a derivadas del tensor G_{ij}.

$$\frac{1}{\left\{g_{\gamma\delta}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds}\right\}^{5/2}}\left[\frac{\partial G_{pq}}{\partial x^\alpha} g_{uv} g_{wt} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} - 2\frac{\partial G_{p\alpha}}{\partial x^q} g_{uv} g_{wt} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} + \frac{\partial G_{pq}}{\partial x^v} g_{u\alpha} g_{wt} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{d\cancel{x}^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds}\right]$$

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PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES. (Segundo miembro = 0)

Términos que contienen a las derivadas del tensor g_{ij}

$$\frac{1}{\left\{g_{\gamma\delta}\, dx^\gamma dx^\delta\right\}^{5/2}}\left[-\frac{1}{2} G_{pq} \frac{\partial g_{uv}}{\partial x^\alpha} g_{wt} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} + G_{p\alpha} \frac{\partial g_{uv}}{\partial x^q} g_{wt} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} + G_{pq} \frac{\partial g_{u\alpha}}{\partial x^v} g_{wt} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} - \frac{3}{2} G_{pq} \frac{\partial g_{uv}}{\partial x^w} g_{\alpha t} \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds}\right]$$

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Definición de las métricas auxiliares:

$$\boxed{d\sigma^2 = g_{ij}\, dx^i dx^j}$$

$$\boxed{dS'^2 = G_{ij}\, dx^i dx^j}$$

$$\boxed{\left(\frac{d\sigma}{ds}\right)^2 = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}}$$

$$\boxed{\left(\frac{dS'}{ds}\right)^2 = G_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}}$$

Símbolos de Christoffel de Primera Especie de:

métrica: $d\sigma^2 = g_{ij}\, dx^i dx^j$;

$$\gamma_{ij,k} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right]$$

métrica $dS^2 = G_{ij}\, dx^i dx^j$

$$\Gamma_{ij,k} = \frac{1}{2}\left[\frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k}\right]$$

Primer miembro de la ecuación de las extremales. (Segundo miembro $= 0$)
Términos que contienen a las segundas derivadas de las coordenadas con respecto a s.

$$\left[-2G_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^4 + 2G_{\alpha p}\, g_{\beta q}\left(\frac{d\sigma}{ds}\right)^2 \frac{dx^p}{ds}\frac{dx^q}{ds} + 2\, g_{\alpha q}\, G_{\beta p}\left(\frac{d\sigma}{ds}\right)^2 \frac{dx^p}{ds}\frac{dx^q}{ds} - 3\, g_{\alpha u}\, g_{\beta v}\left(\frac{dS}{ds}\right)^2 \frac{dx^u}{ds}\frac{dx^v}{ds} + g_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dS}{ds}\right)^2 \right]\frac{d^2x^\beta}{ds^2}$$

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Primer miembro de la ecuación de las extremales (Segundo miembro = 0)

Términos que contienen a las derivadas del Tensor G_{ij}

$$\boxed{\left[-2\Gamma_{pq,\alpha}\left(\frac{d\sigma}{ds}\right)^4\frac{dx^p}{ds}\frac{dx^q}{ds}+2g_{q\nu}\frac{dx^\nu}{ds}\Gamma_{pq,\nu}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^\nu}{ds}\left(\frac{d\sigma}{ds}\right)\right]^2}$$

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<u>Primer miembro de la ecuación de las extremales.</u> (Segundo miembro $= 0$)

<u>Términos que contienen a las derivadas del tensor g_{ij}.</u>

$$\Bigg[+ \left(\frac{dS}{ds}\right)^2 \left(\frac{d\sigma}{ds}\right)^2 \gamma_{uv,\alpha} \frac{dx^u}{ds} \frac{dx^v}{ds}$$

$$+ 2 G_{\alpha\beta} \frac{dx^\beta}{ds} \left(\frac{d\sigma}{ds}\right)^2 \gamma_{uv,q} \frac{dx^u}{ds} \frac{dx^v}{ds} \frac{dx^q}{ds}$$

$$- 3 g_{\alpha t} \frac{dx^t}{ds} \left(\frac{dS}{ds}\right)^2 \gamma_{uv,w} \frac{dx^u}{ds} \frac{dx^v}{ds} \frac{dx^w}{ds} \Bigg]$$

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Definición de los símbolos de Christoffel de segunda especie de las métricas $d\sigma^2$ y dS^2.

$$\left.\begin{aligned}&\boxed{\gamma_{ij,k} = g_{lk}\,\gamma^{l}_{ij}}\\[4pt]&\boxed{|g_{lk}| \neq 0}\end{aligned}\right\}\ d\sigma^2$$

$$\left.\begin{aligned}&\Gamma_{ij,k} = G_{lk}\,\Gamma^{l}_{ij}\\[4pt]&|G_{lk}| \neq 0\,.\end{aligned}\right\}\ dS^2$$

EXTREMALES.

$$0 = \left[-2\,G_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^4 + 2\,G_{\alpha p}\,g_{\beta q}\left(\frac{d\sigma}{ds}\right)^2 \frac{dx^p}{ds}\frac{dx^q}{ds} + 2\,g_{\alpha q}\,G_{\beta p}\left(\frac{d\sigma}{ds}\right)^2 \frac{dx^p}{ds}\frac{dx^q}{ds} - 3\,g_{\alpha u}\,g_{\beta v}\left(\frac{dS}{ds}\right)^2 \frac{dx^u}{ds}\frac{dx^v}{ds} + g_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dS}{ds}\right)^2 \right] \frac{d^2x^\beta}{ds^2}$$

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$$+ \left[-2\,G_{\alpha u}\,\Gamma^{u}_{pq}\left(\frac{d\sigma}{ds}\right)^4 \frac{dx^p}{ds}\frac{dx^q}{ds} + 2\,g_{\alpha u}\,G_{vw}\,\Gamma^{w}_{pq}\left(\frac{d\sigma}{ds}\right)^2 \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds} \right]$$

$$+ \left[g_{\alpha w}\,\gamma^{w}_{uv}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dS}{ds}\right)^2 \frac{dx^u}{ds}\frac{dx^v}{ds} + 2\,g_{qw}\,G_{\alpha p}\,\gamma^{w}_{uv}\left(\frac{d\sigma}{ds}\right)^2 \frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds} \right]$$

$$-3\, g_{\alpha p}\, g_{qw}\, \gamma^{w}_{uv} \left(\frac{\hat{d}s'}{ds}\right)^{2} \frac{dx^{p}}{ds} \frac{dx^{q}}{ds} \frac{dx^{u}}{ds} \frac{dx^{v}}{ds} \Big]$$

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REVISION GENERAL

$$\boxed{\delta\{G_{ij}\,dx^i\,dx^j\}}$$

$$\boxed{\delta\{G_{ij}\,dx^i\,dx^j\} = \frac{\partial G_{ij}}{\partial x^k}\,\delta x^k\,dx^i\,dx^j + 2G_{ij}\,dx^i\,d(\delta x^j)}$$

Comprobado página 3

$$\boxed{\delta\left\{g_{pq}\,dx^p dx^q\right\}^{-\frac{1}{2}}}$$

$$\boxed{= -\frac{1}{2}\left\{g_{pq}\,dx^p dx^q\right\}^{-\frac{3}{2}}\left\{\frac{\partial g_{pq}}{\partial x^r}\,\delta x^r dx^p dx^q + 2 g_{pq}\,dx^p d(\delta x^q)\right\}}$$

comprobado página 4

$$\boxed{\delta\left\{\frac{G_{ij}\,dx^i dx^j}{\sqrt{g_{pq}\,dx^p dx^q}}\right\}}$$

$$\boxed{\begin{aligned}&\left\{g_{pq}\,dx^p dx^q\right\}^{-\frac{1}{2}}\left\{\frac{\partial G_{ij}}{\partial x^k}\delta x^k dx^i dx^j + 2\,G_{ij}\,dx^i\,d(\delta x^j)\right\}\\ &-\frac{1}{2}\left\{g_{pq}\,dx^p dx^q\right\}^{-\frac{3}{2}} G_{ij}\,dx^i dx^j\left\{\frac{\partial g_{pq}}{\partial x^r}\delta x^r dx^p dx^q + 2\,g_{pq}\,dx^p\,d(\delta x^q)\right\}\end{aligned}}$$

Comprobado pág. 5

$$\boxed{\delta\left\{\frac{G_{ij}\,dx^i\,dx^j}{\sqrt{g_{pq}\,dx^p\,dx^q}}\right\}}$$

$$\boxed{\begin{aligned}
&\{g_{pq}\,dx^p\,dx^q\}^{-\frac{1}{2}}\frac{\partial G_{ij}}{\partial x^k}\,\delta x^k\,dx^i\,dx^j\\
&-\frac{1}{2}\{g_{pq}\,dx^p\,dx^q\}^{-\frac{3}{2}}G_{ij}\frac{\partial g_{pq}}{\partial x^r}\,\delta x^r\,dx^i\,dx^j\,dx^p\,dx^q\\
&+2\{g_{pq}\,dx^p\,dx^q\}^{-\frac{1}{2}}G_{ij}\,dx^i\,d(\delta x^j)\\
&-\{g_{pq}\,dx^p\,dx^q\}^{-\frac{3}{2}}G_{ij}\,g_{pq}\,dx^i\,dx^j\,dx^p\,d(\delta x^q)
\end{aligned}}$$

$$\boxed{\delta\left\{\frac{G_{ij}\,dx^i\,dx^j}{\sqrt{g_{pq}\,dx^p\,dx^q}}\right\}}$$

$$\boxed{\begin{aligned}&\{g_{pq}\,dx^p dx^q\}^{-\frac{1}{2}}\,\frac{\partial G_{ij}}{\partial x^\alpha}\,\delta x^\alpha\,dx^i\,dx^j\\ &-\tfrac{1}{2}\{g_{pq}\,dx^p dx^q\}^{-\frac{3}{2}}\,G_{ij}\,\frac{\partial g_{pq}}{\partial x^\alpha}\,\delta x^\alpha\,dx^i\,dx^j\,dx^p\,dx^q\\ &+2\{g_{pq}\,dx^p dx^q\}^{-\frac{1}{2}}\,G_{i\alpha}\,dx^i\,d(\delta x^\alpha)\\ &-\{g_{pq}\,dx^p dx^q\}^{-\frac{3}{2}}\,G_{ij}\,g_{p\alpha}\,dx^i\,dx^j\,dx^p\,d(\delta x^\alpha)\end{aligned}}$$

$$\boxed{\delta\left\{\frac{G_{ij}\,dx^i\,dx^j}{\sqrt{g_{pq}\,dx^p\,dx^q}}\right\}}$$

$$+\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{1}{2}}\frac{\partial G_{ij}}{\partial x^\alpha}\,\delta x^\alpha\,\frac{dx^i}{ds}\frac{dx^j}{ds}\,ds$$

$$-\frac{1}{2}\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}}G_{ij}\,\frac{\partial g_{pq}}{\partial x^\alpha}\,\delta x^\alpha\,\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\,ds$$

$$+2\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{1}{2}}G_{i\alpha}\frac{dx^i}{ds}\,d(\delta x^\alpha)$$

$$-\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}}G_{ij}\,g_{p\alpha}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\,d(\delta x^\alpha)$$

Comprobado página 6.

$$\int_A^B \Bigg[-2\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{1}{2}}\frac{\partial G_{i\alpha}}{\partial x^j}\frac{dx^i}{ds}\frac{dx^j}{ds}\,\delta x^\alpha\, ds$$

$$-2\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{1}{2}} G_{i\alpha}\frac{d^2x^i}{ds^2}\,\delta x^\alpha\, ds$$

$$+\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} G_{i\alpha}\frac{dx^i}{ds}\frac{\partial g_{pq}}{\partial x^j}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^j}{ds}\,\delta x^\alpha\, ds$$

$$+2\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} G_{i\alpha}\frac{dx^i}{ds}\, g_{pq}\frac{d^2x^p}{ds^2}\frac{dx^q}{ds}\cancel{\frac{dx^i}{ds}}\,\delta x^\alpha\, ds \Bigg]$$

Comprobado página 7

$$\boxed{+2\int_A^B \left\{ g_{pq} \frac{dx^p}{ds} \frac{dx^q}{ds} \right\}^{-\frac{1}{2}} G_{i\alpha} \frac{dx^i}{ds}\, d(\delta x^\alpha)}$$

$$-2 \left\{ g_{pq} \frac{dx^p}{ds} \frac{dx^q}{ds} \right\}^{-\frac{1}{2}} \frac{\partial G_{i\alpha}}{\partial x^j} \frac{dx^i}{ds} \frac{dx^j}{ds} \delta x^\alpha\, d$$

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$$\boxed{-\int_A^B \left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} G_{ij}\, g_{p\alpha}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\, d(\delta x^\alpha)}$$

$$\int_A^B \Bigg[+ \cancel{\frac{\partial G_{ij}}{\partial x^q} g_{p\alpha}} \left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} \frac{\partial G_{ij}}{\partial x^q} g_{p\alpha}\,\delta x^\alpha \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\,ds$$

$$+\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} G_{ij}\,\frac{\partial g_{p\alpha}}{\partial x^q}\,\delta x^\alpha \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\,ds$$

$$+2\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} G_{ij}\, g_{p\alpha}\,\delta x^\alpha \frac{d^2x^i}{ds^2}\frac{dx^j}{ds}\frac{dx^p}{ds}\,ds$$

$$+\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{3}{2}} G_{ij}\, g_{p\alpha}\,\delta x^\alpha \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{d^2x^p}{ds^2}\,ds$$

$$-\frac{3}{2}\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{5}{2}} G_{ij}\, g_{p\alpha}\,\frac{\partial g_{\nu q}}{\partial x^\nu}\,\delta x^\alpha \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\frac{dx^\nu}{ds}\frac{dx^q}{ds}\,ds$$

$$-\frac{3}{2}\left\{g_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right\}^{-\frac{5}{2}} G_{ij}\, g_{p\alpha}\, g_{\nu q}\,\delta x^\alpha \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{d^2x^\nu}{ds^2}\,ds \Bigg]$$

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Comprobado página 6

$$-\int_A^B \left\{ g_{pq} \frac{dx^p}{ds} \frac{dx^q}{ds} \right\}^{-\frac{3}{2}} G_{ij}\, g_{p\alpha} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^p}{ds}\, d$$

$$+ \frac{\partial G_{ij}}{\partial x^{\alpha}} g_{p\alpha} \left\{ g_{pq} \frac{dx^p}{ds} \frac{dx^q}{ds} \right\}^{-\frac{1}{2}} \frac{\partial G_{ij}}{\partial x^q} g_{p\alpha}$$

$$+ \left\{ g_{pq} \frac{dx^p}{ds} \frac{dx^q}{ds} \right\}^{-\frac{1}{2}} G_{ij} \frac{\partial g_{p\alpha}}{\partial x^{\alpha}} \frac{dx^i}{ds} \frac{dx}{}$$

$$\frac{1}{\left(g_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\right)^{5/2}}\left[\begin{aligned}&-2G_{\alpha\beta}\,g_{pq}\,g_{uv}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\\&+2G_{\alpha p}\,g_{\beta q}\,g_{uv}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\\&+2G_{\beta p}\,g_{\alpha q}\,g_{uv}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\\&-3G_{pq}\,g_{\alpha u}\,g_{\beta v}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\\&+G_{pq}\,g_{\alpha\beta}\,g_{uv}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\end{aligned}\right]\frac{d^2x^\beta}{ds^2}$$

Comprobado página 10.

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES,

(Segundo miembro = 0) Términos que contienen a las segundas derivadas de las coordenadas)

$\Big[\quad dx^\rho dx^q \; dx^\mu \; dx^\nu \quad \Big]^{\rho\beta}$

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES.

(Segundo miembro = 0) Términos que contienen a las derivadas del tensor G_{ij}

$$\frac{1}{\left\{g_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\right\}^{5/2}}\left[+\frac{\partial G_{pq}}{\partial x^\alpha}g_{uv}g_{wt}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} - 2\frac{\partial G_{\alpha p}}{\partial x^q}g_{uv}g_{wt}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds} + \frac{\partial G_{pq}}{\partial x^u}g_{\alpha v}g_{wt}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds}\right]$$

Comprobado en la página 10 (11)

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES.

(Segundo miembro = 0) Términos que contienen a las derivadas del tensor G_{ij}

... las derivadas del Tensor g_{uv}

$$\frac{1}{\left\{g_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\right\}^{5/2}}\left[-\frac{1}{2}G_{pq}\frac{\partial g_{uv}}{\partial x^\alpha}g_{wt}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{dt}\frac{dx^t}{ds}\right.$$

$$+G_{\alpha p}\frac{\partial g_{qu}}{\partial x^v}g_{wt}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds}$$

$$+G_{pq}\frac{\partial g_{\alpha u}}{\partial x^v}g_{wt}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds}$$

$$\left.-\frac{3}{2}G_{pq}\frac{\partial g_{uv}}{\partial x^w}g_{\alpha t}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^u}{ds}\frac{dx^v}{ds}\frac{dx^w}{ds}\frac{dx^t}{ds}\right]$$

Comprobado página 11'

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES, (Segundo miembro = 0) Términos que contienen a las derivadas del tensor g_{ii}

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES.

[Segundo miembro = 0] Después de multiplicar las expresiones de las páginas 29, 30 y 31 por $\left\{g_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}\right\}^{5/2}$.

Definición de dos métricas auxiliares

$d\sigma^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \quad \| \quad dS^2 = G_{pq}\, dx^p dx^q$

~~Términos que cont~~

Términos que contienen a las segundas derivadas de las coordenadas:

de las coordenadas,

$$\left[-2\,G_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^4 + 2\,G_{\alpha p}\,g_{\beta q}\,\frac{dx^p}{ds}\frac{dx^q}{ds}\left(\frac{d\sigma}{ds}\right)^2 + 2\,G_{\beta p}\,g_{\alpha q}\,\frac{dx^p}{ds}\frac{dx^q}{ds}\left(\frac{d\sigma}{ds}\right)^2 - 3\,g_{\alpha p}\,g_{\beta q}\left(\frac{dS}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds} + g_{\alpha\beta}\left(\frac{dS}{ds}\right)^2\left(\frac{d\sigma}{ds}\right)^2\right]\frac{d^2x^\beta}{ds^2}$$

Comprobado página 15

$$\left[-2\,G_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^4 + 2\,G_{\alpha p}\,g_{\beta q}\left(\frac{d\sigma}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds} + 2\,G_{\beta p}\,g_{\alpha q}\left(\frac{d\sigma}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds} - 3\,g_{\alpha p}\,g_{\beta q}\left(\frac{d\delta}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds} + g_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{d\delta}{ds}\right)^2\right]\frac{d^2x^\beta}{ds^2}$$

Comprobado pág. 15

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES,

[Segundo miembro = 0] Después de multiplicar las expresiones de las páginas 29, 30 ~~y~~ 31 por $d\sigma^5$. Métricas auxiliares $d\sigma^2 = g_{ij}\, dx^i dx^j$, $ds^2 = G_{pq}\, dx^p dx^q$.

Términos que contienen a las segundas derivadas de las coordenadas con respecto a s. Ver página 32

Definición de las 2 métricas auxiliares:

$$\boxed{\begin{aligned} d\sigma^2 &= g_{ij}\, dx^i dx^j \,; \\ d\sigma'^2 &= G_{pq}\, dx^p dx^q . \end{aligned}}$$

Símbolos de Christoffel de primera especie

Primera métrica:

$$d\sigma^2 = g_{ij}\, dx^i dx^j$$

$$\gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Segunda métrica:

$$ds^2 = G_{pq}\,dx^p dx^q$$

$$\Gamma_{pq,r} = \frac{1}{2}\left[\frac{\partial G_{pr}}{\partial x^q} + \frac{\partial G_{qr}}{\partial x^p} - \frac{\partial G_{pq}}{\partial x^r}\right]$$

Definición de los símbolos de Christoffel de Segunda Especie.

Primera métrica.

$$g_{iu}\,\gamma^i_{jk} = \gamma_{jk,u}$$

Segunda métrica.

$$G_{pq}\,\Gamma^q_{rs} = \Gamma_{rs,p}$$

multiplicada por $d\sigma^5$.

$$\left[-2\Gamma_{pq,\alpha}\left(\frac{d\sigma}{ds}\right)^4\frac{dx^p}{ds}\frac{dx^q}{ds}+2g_{\alpha\nu}\Gamma_{pq,\upsilon}\frac{dx^\nu}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^\upsilon}{ds}\left(\frac{d\sigma}{ds}\right)^2\right]$$

Comprobado página 16.

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES. (segundo miembro $=0$)

Términos que contienen a las derivadas de G_{ij}. La expresión de la página 30 multiplicada por $d\sigma^5$.

$$\Big[+ \gamma_{uv,\alpha}\left(\frac{dS}{ds}\right)^2\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dx^u}{ds}\right)\left(\frac{dx^v}{ds}\right) - 3\gamma_{uw,v}\, g_{\alpha t}\left(\frac{dS}{ds}\right)^2 \frac{dx^u}{ds}\frac{dx^w}{ds}\frac{dx^v}{ds}\frac{dx^t}{ds}$$

$$+ 2\gamma_{qv,u}\,\Theta_{\alpha p}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dx^q}{ds}\right)\left(\frac{dx^v}{ds}\right)\left(\frac{dx^u}{ds}\right)\left(\frac{dx^p}{ds}\right)\Big]$$

comprobado página 17

PRIMER MIEMBRO DE LA ECUACION DE LAS EXTREMALES. (Segundo miembro = 0) Términos que contienen a las derivadas del tensor g_{ij}.

Expresión de la página 31 multiplicada por $d\sigma^5$.

EXTREMALES. $0=$

$$\left[-2\,G_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^4 + \boxed{g_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dS}{ds}\right)^2} + 2\left(G_{\alpha p}\,g_{\beta q} + G_{\beta p}\,g_{\alpha q}\right)\left(\frac{d\sigma}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds} - 3\,g_{\alpha p}\,g_{\beta q}\left(\frac{dS}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds}\right]\frac{d^2x^\beta}{ds^2}$$

$$+\left[-2\,G_{\alpha\nu}\,\Gamma^{\nu}_{pq}\left(\frac{d\sigma}{ds}\right)^4\frac{dx^p}{ds}\frac{dx^q}{ds} + 2\,g_{\alpha\nu}\,G_{\upsilon w}\,\Gamma^{w}_{pq}\left(\frac{d\sigma}{ds}\right)^2\frac{dx^p}{ds}\frac{dx^q}{ds}\frac{dx^\upsilon}{ds}\frac{dx^\nu}{ds}\right]$$

$$+\left[\boxed{g_{\alpha w}\,\gamma^{w}_{\upsilon\nu}\left(\frac{dS}{ds}\right)^2\left(\frac{d\sigma}{ds}\right)^2\frac{dx^\upsilon}{ds}\frac{dx^\nu}{ds}} - 3\,g_{\nu\rho}\,\gamma^{\rho}_{\upsilon w}\,g_{\alpha t}\left(\frac{dS}{ds}\right)^2\frac{dx^\upsilon}{ds}\frac{dx^w}{ds}\frac{dx^\nu}{ds}\frac{dx^t}{ds}\right.$$

$$\left.+2\,G_{\alpha p}\,g_{\upsilon w}\,\gamma^{w}_{q\nu}\left(\frac{d\sigma}{ds}\right)^2\frac{dx^\upsilon}{ds}\frac{dx^\nu}{ds}\frac{dx^p}{ds}\frac{dx^q}{ds}\right]$$

Ecuación diferencial de una geodésica en términos de un parámetro τ.

$$\boxed{\delta J = \delta \int_A^B \sqrt{g_{ij}\, dx^i dx^j} = 0}$$

$$\delta J = \int_A^B \frac{\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\delta x^k dx^i dx^j + g_{ij}\, dx^i\, d(\delta x^j)}{\sqrt{g_{ij}\, dx^i dx^j}}$$

$$\boxed{J = \int_A^B \sqrt{g_{ij}\,dx^i\,dx^j}\ .}$$

$$\delta\left\{\sqrt{g_{ij}\,dx^i\,dx^j}\right\} = \frac{\frac{\partial g_{ij}}{\partial x^k}\,\delta x^k\,dx^i\,dx^j}{2\sqrt{g_{pq}\,dx^p\,dx^q}} + \frac{g_{ij}\,dx^i\,d(\delta x^j)}{\sqrt{g_{pq}\,dx^p\,dx^q}}$$

$$\boxed{\delta J = \int_A^B \frac{\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}{2\sqrt{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}}}\,\delta x^k\,d\tau + \int_A^B \frac{g_{ij}\frac{dx^i}{d\tau}\,d(\delta x^j)}{\sqrt{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}}}}$$

$$\delta J = (\delta J)_1 + (\delta J)_2\ .$$

$$\boxed{(\delta J)_1 = \int_A^B \frac{\dfrac{\partial g_{ij}}{\partial x^k} \dfrac{dx^i}{d\tau} \dfrac{dx^j}{d\tau}}{2\sqrt{g_{pq} \dfrac{dx^p}{d\tau} \dfrac{dx^q}{d\tau}}} \delta x^k \, d\tau}$$

$$\boxed{(\delta J)_2 = \int_A^B \frac{g_{ij} \dfrac{dx^i}{d\tau} \, d(\delta x^j)}{\sqrt{g_{pq} \dfrac{dx^p}{d\tau} \dfrac{dx^q}{d\tau}}}}$$

$$\left(\delta J\right)_2 = \int_A^B \left\{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right\}^{-\frac{1}{2}} g_{ij}\frac{dx^i}{d\tau}\, d\left(\delta x^j\right)$$

$$\left(\delta J\right)_2 = \int_A^B \left[+\frac{1}{2}\left\{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right\}^{-\frac{3}{2}} g_{ij}\frac{dx^i}{d\tau}\frac{\partial g_{pq}}{\partial x^r}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\frac{dx^r}{d\tau} + \left\{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right\}^{-\frac{3}{2}} g_{ij}\frac{dx^i}{d\tau}\, g_{pq}\frac{d^2x^p}{d\tau^2}\frac{dx^q}{d\tau} - \left\{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right\}^{-\frac{1}{2}} \frac{\partial g_{ij}}{\partial x^r}\frac{dx^i}{d\tau}\frac{dx^r}{d\tau} - \left\{g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\right\}^{-\frac{1}{2}} g_{ij}\frac{d^2x^i}{d\tau^2} \right] \delta x^j\, d\tau$$

Ecuación geodésicas

$$
\left.
\begin{aligned}
&+ g_{ij} \frac{dx^i}{d\tau} \cancel{\frac{dx^j}{d\tau}} g_{pq} \frac{d^2x^p}{d\tau^2} \frac{dx^q}{d\tau} - g_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} g_{ij} \frac{d^2x^i}{d\tau^2} \\
&+ \frac{1}{2} g_{ij} \frac{dx^i}{d\tau} \frac{\partial g_{pq}}{\partial x^r} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} \frac{dx^r}{d\tau} \\
&- g_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} \frac{\partial g_{ij}}{\partial x^r} \frac{dx^i}{d\tau} \frac{dx^r}{d\tau} \\
&+ \frac{1}{2} \frac{\partial g_{ir}}{\partial x^j} \frac{dx^i}{d\tau} \frac{dx^r}{d\tau} g_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau}
\end{aligned}
\right\} = 0
$$

Ecuación geodésicas

$$\left.\begin{aligned} & g_{ij}\frac{dx^i}{d\tau}\, g_{pq}\frac{d^2x^p}{d\tau^2}\frac{dx^q}{d\tau} - g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\, g_{ij}\frac{d^2x^i}{d\tau^2} \\ & + g_{ij}\frac{dx^i}{d\tau}\Gamma_{pq,r}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\frac{dx^r}{d\tau} - g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\Gamma_{ir,j}\frac{dx^i}{d\tau}\frac{dx^r}{d\tau} \end{aligned}\right\} = 0$$

$$\Gamma_{pq,r} = g_{rs}\Gamma^s_{pq} \qquad \Gamma_{ir,j} = g_{jt}\Gamma^t_{ir}$$

$$\left.\begin{aligned} & g_{ij}\frac{dx^i}{d\tau}\, g_{pq}\frac{dx^q}{d\tau}\frac{d^2x^p}{d\tau^2} - g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\, g_{ij}\frac{d^2x^i}{d\tau^2} \\ & + g_{ij}\frac{dx^i}{d\tau}\, g_{rs}\frac{dx^r}{d\tau}\Gamma^s_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau} - g_{pq}\frac{dx^p}{d\tau}\frac{dx^q}{d\tau}\, g_{jt}\Gamma^t_{ir}\frac{dx^i}{d\tau}\frac{dx^r}{d\tau} \end{aligned}\right\} = 0$$

multiplíquese g^{iv} $\underline{\underline{g^{iv}}}$

$$\left.\begin{aligned} &\frac{dx^v}{d\tau} g_{pq} \frac{dx^q}{d\tau} \frac{d^2x^p}{d\tau^2} - g_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} \frac{d^2x^v}{d\tau^2} \\ &\frac{dx^v}{d\tau} g_{rs} \frac{dx^r}{d\tau} \Gamma^s_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} - g_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} \Gamma^v_{ir} \frac{dx^i}{d\tau} \frac{dx^r}{d\tau} \end{aligned}\right\} = e$$

$$\left.\begin{aligned} &\frac{dx^v}{d\tau} g_{pq} \frac{dx^q}{d\tau} \left[\frac{d^2x^p}{d\tau^2} + \Gamma^p_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \right] \\ &- g_{pq} \frac{dx^p}{d\tau} \frac{dx^q}{d\tau} \left[\frac{d^2x^v}{d\tau^2} + \Gamma^v_{ir} \frac{dx^i}{d\tau} \frac{dx^r}{d\tau} \right] \end{aligned}\right\} = 0$$

$$g_{pq}\frac{dx^q}{d\tau}\left[\frac{dx^\nu}{d\tau}\left\{\frac{d^2x^p}{d\tau^2}+\Gamma^p_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}\right\} - \frac{dx^p}{d\tau}\left\{\frac{d^2x^\nu}{d\tau^2}+\Gamma^\nu_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}\right\}\right]=0$$

¡Geodésicas!

ECUACION DE LAS GEODESICAS

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$\boxed{\tau = \tau(s)}$$

$$\boxed{\frac{d}{ds} = \frac{d\tau}{ds}\frac{d}{d\tau}}$$

$$\boxed{\frac{d^2}{ds^2} = \frac{d^2\tau}{ds^2}\frac{d}{d\tau} + \left(\frac{d\tau}{ds}\right)^2\frac{d^2}{d\tau^2}}$$

$$\frac{d^2x^i}{d\tau^2}\left(\frac{d\tau}{ds}\right)^2 + \frac{d^2\tau}{ds^2}\frac{dx^i}{d\tau} + \Gamma^i_{jk}\left(\frac{d\tau}{ds}\right)^2\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

$$\boxed{\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = -\boxed{\frac{\frac{d^2\tau}{ds^2}}{\left(\frac{d\tau}{ds}\right)^2}\frac{dx^i}{d\tau}}}$$

CAMBIO DE VARIABLE EN LA ECUACION DE LAS GEODÉSICAS,

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

Transformación de las derivadas de un tensor simétrico doblemente covariante..

$|T_{ij}| \neq 0$ $\quad T_{ij}$ Tensor simétrico. $T_{ij} = T_{ji}$

$$\boxed{T'_{ij} = \frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j} T_{pq}}$$

$$\frac{\partial T'_{ij}}{\partial x'^k} = \left[\frac{\partial^2 x^p}{\partial x'^i \partial x'^k}\frac{\partial x^q}{\partial x'^j} T_{pq} + \frac{\partial x^p}{\partial x'^i}\frac{\partial^2 x^q}{\partial x'^j \partial x'^k} T_{pq} + \frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j}\frac{\partial T_{pq}}{\partial x^r}\frac{\partial x^r}{\partial x'^k} \right]$$

Determinante de $T_{ij} \neq 0$

$$\frac{1}{2}\left[\frac{\partial T'_{ik}}{\partial x'^j} + \frac{\partial T'_{jk}}{\partial x'^i} - \frac{\partial T'_{ij}}{\partial x'^k}\right]$$

$$\frac{1}{2}\left[\frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j}\frac{\partial x^r}{\partial x'^k}\frac{\partial T_{pr}}{\partial x^q} + \frac{\partial x^p}{\partial x'^i}\frac{\partial^2 x^q}{\partial x'^j \partial x'^k} T_{pq} + \frac{\partial x^q}{\partial x'^k}\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} T_{pq}\right.$$

$$+ \frac{\partial x^r}{\partial x'^k}\frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j}\frac{\partial T_{rq}}{\partial x^p} + \frac{\partial x^r}{\partial x'^k}\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} T_{rp} + \frac{\partial x^p}{\partial x'^j}\frac{\partial^2 x^r}{\partial x'^k \partial x'^i} T_{rp}$$

$$\left. - \frac{\partial x^q}{\partial x'^j}\frac{\partial x^r}{\partial x'^k}\frac{\partial x^p}{\partial x'^i}\frac{\partial T_{qp}}{\partial x^r} - \frac{\partial x^q}{\partial x'^j}\frac{\partial^2 x^r}{\partial x'^k \partial x'^i} T_{qr} - \frac{\partial x^r}{\partial x'^i}\frac{\partial^2 x^q}{\partial x'^j \partial x'^k} T_{qr}\right]$$

$$\frac{1}{2}\left[\frac{\partial T'_{ik}}{\partial x'^j} + \frac{\partial T'_{jk}}{\partial x'^i} - \frac{\partial T'_{ij}}{\partial x'^k}\right]$$

$$\frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j}\frac{\partial x^r}{\partial x'^k}\frac{1}{2}\left[\frac{\partial T_{pr}}{\partial x^q} + \frac{\partial T_{rq}}{\partial x^p} - \frac{\partial T_{pq}}{\partial x^r}\right] + \frac{\partial^2 x^p}{\partial x'^i \partial x'^j}\frac{\partial x^q}{\partial x'^k}\, T_{pq}$$

Comprobado página 105.

Definición: $\Theta_{pq,r} = \frac{1}{2}\left[\frac{\partial T_{pr}}{\partial x^q} + \frac{\partial T_{qr}}{\partial x^p} - \frac{\partial T_{pq}}{\partial x^r}\right]$

$$\boxed{\Theta'_{ij,k} = \frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j}\frac{\partial x^r}{\partial x'^k}\,\Theta_{pq,r} + \frac{\partial^2 x^p}{\partial x'^i \partial x'^j}\frac{\partial x^q}{\partial x'^k}\, T_{pq}}$$

Transformación de la expresión

$$T_{am}\frac{d^2x^m}{d\tau^2}+\frac{1}{2}\left[\frac{\partial T_{am}}{\partial x^n}+\frac{\partial T_{an}}{\partial x^m}-\frac{\partial T_{mn}}{\partial x^a}\right]\frac{dx^m}{d\tau}\frac{dx^n}{d\tau}=\Sigma_a$$

$$T'_{am}\frac{d^2x'^m}{d\tau^2}+\frac{1}{2}\left[\frac{\partial T'_{am}}{\partial x'^n}+\frac{\partial T'_{an}}{\partial x'^m}-\frac{\partial T'_{mn}}{\partial x'^a}\right]\frac{dx'^m}{d\tau}\frac{dx'^n}{d\tau}=\Sigma'_a$$

$$\boxed{T'_{am}=\frac{\partial x^p}{\partial x'^a}\frac{\partial x^q}{\partial x'^m}T_{pq}}$$

$$\frac{dx'^m}{d\tau}=\frac{\partial x'^m}{\partial x^c}\frac{dx^c}{d\tau}$$

$$\boxed{\frac{d^2x'^m}{d\tau^2}=\frac{\partial x'^m}{\partial x^c}\frac{d^2x^c}{d\tau^2}+\frac{\partial^2 x'^m}{\partial x^c\partial x^d}\frac{dx^c}{d\tau}\frac{dx^d}{d\tau}}$$

$$\frac{\partial x^p}{\partial x'^a}\frac{\partial x^q}{\partial x'^m} T_{pq} \frac{\partial x'^m}{\partial x^c}\frac{d^2 x^c}{d\tau^2} + \frac{\partial x^p}{\partial x'^a}\frac{\partial x^q}{\partial x'^m} T_{pq} \frac{\partial^2 x'^m}{\partial x^c \partial x^d}\frac{dx^c}{d\tau}\frac{dx^d}{d\tau}$$

$$+ \frac{\partial x^p}{\partial x'^m}\frac{\partial x^q}{\partial x'^n}\frac{\partial x^r}{\partial x'^a}\Theta_{pq,r}\frac{\partial x'^m}{\partial x^s}\frac{\partial x'^n}{\partial x^t}\frac{dx^s}{d\tau}\frac{dx^t}{d\tau}$$

$$+ \frac{\partial^2 x^{p \leftarrow q}}{\partial x'^m \partial x'^n}\frac{\partial x^{q \leftarrow p}}{\partial x'^a} T_{pq} \frac{\partial x'^m}{\partial x^s}\frac{\partial x'^n}{\partial x^t}\frac{dx^s}{d\tau}\frac{dx^t}{d\tau}$$

$$\frac{\partial x^p}{\partial x'^a} T_{pc}\frac{d^2 x^c}{d\tau^2} + \frac{\partial x^r}{\partial x'^a}\Theta_{st,r}\frac{dx^s}{dt}\frac{dx^t}{dt}$$

$$+ \frac{\partial x^p}{\partial x'^a} T_{pq}\left[\frac{\partial x^q}{\partial x'^m}\frac{\partial^2 x'^m}{\partial x^c \partial x^d}\frac{dx^c}{d\tau}\frac{dx^d}{d\tau} + \frac{\partial^2 x^q}{\partial x'^m \partial x'^n}\frac{\partial x'^m}{\partial x^s}\frac{\partial x'^n}{\partial x^t}\frac{dx^s}{d\tau}\frac{dx^t}{d\tau}\right]$$

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$$T'_{am} \frac{d^2 x'^m}{d\tau^2} + \Theta'_{mn,a} \frac{dx'^m}{d\tau} \frac{dx'^n}{d\tau}$$

$$\boxed{T'_{am} \frac{d^2 x'^m}{d\tau^2} + \frac{1}{2}\left[\frac{\partial T'_{am}}{\partial x'^n} + \frac{\partial T'_{an}}{\partial x'^m} - \frac{\partial T'_{mn}}{\partial x'^a}\right] \frac{dx'^m}{d\tau} \frac{dx'^n}{d\tau}}$$

$$\frac{\partial x^p}{\partial x'^a}\left[T_{pc}\frac{d^2x^c}{d\tau^2} + \textcircled{H}_{st,p}\frac{dx^s}{d\tau}\frac{dx^t}{d\tau}\right]$$

$$+\frac{\partial x^p}{\partial x'^a}\frac{dx^c}{d\tau}\frac{dx^d}{d\tau}T_{pq}\left[\frac{\partial x^q}{\partial x'^m}\frac{\partial^2 x'^m}{\partial x^c\,\partial x^d} + \frac{\partial^2 x^q}{\partial x'^m\,\partial x'^n}\frac{\partial x'^m}{\partial x^c}\frac{\partial x'^n}{\partial x^d}\right]$$

$$\boxed{\frac{\partial x^q}{\partial x'^m}\frac{\partial x'^m}{\partial x^c} = \delta^q_c}$$

$$\boxed{\frac{\partial}{\partial x^d}}$$

$$\frac{\partial^2 x^q}{\partial x'^m \partial x'^n}\frac{\partial x'^n}{\partial x^d}\frac{\partial x'^m}{\partial x^c} + \frac{\partial x^q}{\partial x'^m}\frac{\partial^2 x'^m}{\partial x^c \partial x^d} = 0$$

$$T'_{am} \frac{d^2 x'^m}{d\tau^2} + \Theta'_{mn,a} \frac{dx'^m}{d\tau} \frac{dx'^n}{d\tau}$$

$$= \frac{\partial x^p}{\partial x'^a} \left[T_{pc} \frac{d^2 x^c}{d\tau^2} + \Theta_{st,p} \frac{dx^s}{d\tau} \frac{dx^t}{d\tau} \right]$$

$$\left. \underline{\underline{T_{pc} \frac{d^2 x^c}{d\tau^2} + \Theta_{st,p} \frac{dx^s}{d\tau} \frac{dx^t}{d\tau}}} \right\} \text{es un vector covariante}$$

RESULTADO

$$\cancel{T_{pk}\frac{d^2x^k}{d\tau^2} + H_{im,k}}$$

$$\boxed{T_{pk}\frac{d^2x^p}{d\tau^2} + H_{ij,k}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = \Sigma_k}$$ Vector Covariante

RESULTADO

T_{pq} es un tensor simétrico doblemente covariante.

$$H_{ij,k} = \frac{1}{2}\left[\frac{\partial T_{ik}}{\partial x^j} + \frac{\partial T_{jk}}{\partial x^i} - \frac{\partial T_{ij}}{\partial x^k}\right];$$

$$T_{pk}\frac{d^2x^p}{d\tau^2} + H_{ij,k}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = \Sigma_k.$$

Σ_k es un vector covariante.

T_{pq} es un tensor doblemente covariante.

$$S^{pq} = \frac{\text{Cofactor de } T_{qp} \text{ en } |T|}{|T|}$$

$$\cancel{S^{iq}}$$

$$S^{ij}\, T_{jk} = \delta^{i}_{k} \leftarrow \text{¡Tensor!}$$

Tensor

S^{ij} — tensor.

$$S'^{ij} \frac{\partial x^p}{\partial x'^j} T_{pr} = \frac{\partial x'^i}{\partial x^r}$$

$$S'^{ij} \frac{\partial x^p}{\partial x'^j} \frac{\partial x^t}{\partial x'^i} T_{pr} = \frac{\partial x'^i}{\partial x^r} \frac{\partial x^t}{\partial x'^i}$$

$$\boxed{S'^{ij} \frac{\partial x^p}{\partial x'^j} \frac{\partial x^t}{\partial x'^i}} T_{pr} = \delta^t_r$$

$$\cancel{S^{pt} = S'}$$

$$S^{tp} = S'^{ij} \frac{\partial x^p}{\partial x'^j} \frac{\partial x^t}{\partial x'^i}$$

Continuación de los desarrollos de la página 38.

$$\left.\begin{aligned}
&-2G_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^4\left[\frac{d^2x^\beta}{ds^2}+\Gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right] && M\\
&+g_{\alpha\beta}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dS}{ds}\right)^2\left[\frac{d^2x^\beta}{ds^2}+\gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right] && m\\
&-3g_{\alpha\upsilon}g_{\beta\nu}\frac{dx^\upsilon}{ds}\frac{dx^\nu}{ds}\left(\frac{dS}{ds}\right)^2\left[\frac{d^2x^\beta}{ds^2}+\gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right] && m\\
&+2g_{\alpha\upsilon}G_{\beta\nu}\frac{dx^\upsilon}{ds}\frac{dx^\nu}{ds}\left(\frac{d\sigma}{ds}\right)^2\left[\frac{d^2x^\beta}{ds^2}+\Gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right] && M\\
&+2G_{\alpha\upsilon}g_{\beta\nu}\frac{dx^\upsilon}{ds}\frac{dx^\nu}{ds}\left(\frac{d\sigma}{ds}\right)^2\left[\frac{d^2x^\beta}{ds^2}+\gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}\right] && m
\end{aligned}\right\}=0$$

INTRODUCCION DE LOS VECTORES $\hat{w}$ y $\hat{W}$.

$$\hat{w}^\beta = \frac{d^2x^\beta}{ds^2} + \gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}$$

$$\boxed{\hat{W}^\beta = \frac{d^2x^\beta}{ds^2} + \Gamma^\beta_{pq}\frac{dx^p}{ds}\frac{dx^q}{ds}}$$

EXTREMALES

$$\left[-2\,\epsilon_{\alpha\rho}\left(\frac{d\sigma}{ds}\right)^4 + 2\,g_{\alpha\nu}\,\epsilon_{\beta\nu}\frac{dx^\nu}{ds}\frac{dx^\nu}{ds}\left(\frac{d\sigma}{ds}\right)^2\right]W^\beta$$

$$+\left[g_{\alpha\rho}\left(\frac{d\sigma}{ds}\right)^2\left(\frac{dS}{ds}\right)^2 - 3\,g_{\alpha\nu}\,g_{\beta\nu}\frac{dx^\nu}{ds}\frac{dx^\nu}{ds}\left(\frac{dS}{ds}\right)^2 + 2\,\epsilon_{\alpha\nu}\cdot g_{\beta\nu}\frac{dx^\nu}{ds}\frac{dx^\nu}{ds}\left(\frac{d\sigma}{ds}\right)^2\right]W^\beta$$